

Gröbner bases and Lefschetz properties

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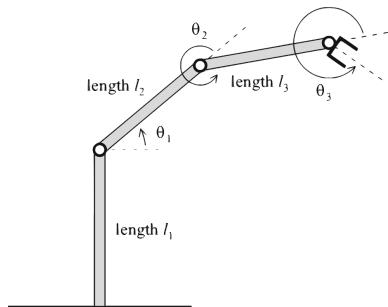
Team Day Department of Mathematics
Leuven, 6 June 2025

Acknowledgments

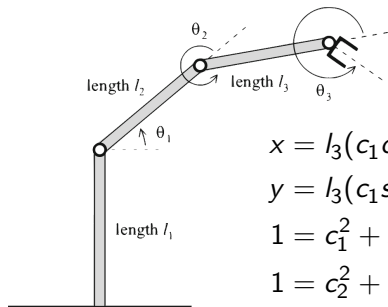
This talk is based on joint work with

- Clemens Hofstadler (Johannes Kepler University Linz),
- Filip Jonsson Kling and Samuel Lundqvist (University of Stockholm),
- Fatemeh Mohammadi (KU Leuven),
- Eduardo Sáenz de Cabezón (Universidad de La Rioja).

Polynomial equations and Gröbner bases



Polynomial equations and Gröbner bases



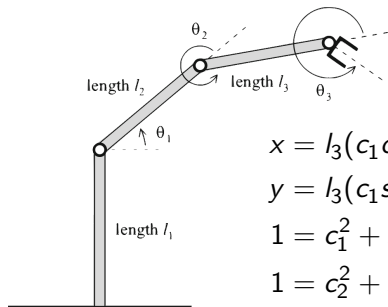
$$x = l_3(c_1 c_2 - s_1 s_2) + l_2 c_1$$

$$y = l_3(c_1 s_2 + c_2 s_1) + l_2 s_1$$

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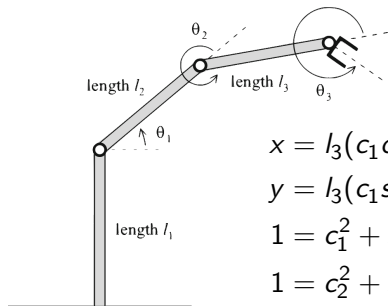
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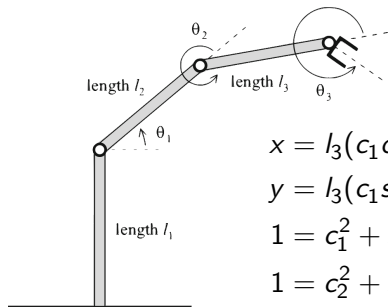
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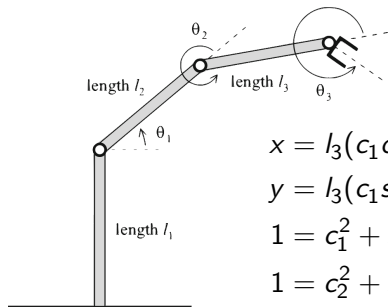
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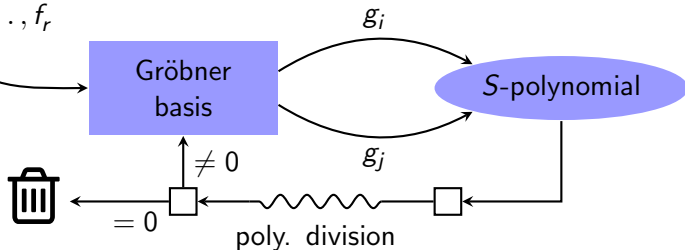
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poly. $f_1, \dots, f_r$



All Gröbner bases of all ideals of a certain form

Theorem (JLMOS 2024)

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Consider the family $\mathcal{A}$ of subsets $A \subseteq [n]$ satisfying $\max(A) = 2|A| - k$ for some $k \geq 2$, and minimal with respect to inclusion.

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where for $|A| = d$,

$$g_{A,n,k} = e_d(x_{i_1}, \dots, x_{i_{n-d+k}})$$

is the elementary symmetric polynomial of degree d in the variables indexed by the set $\{i_1, \dots, i_{n-d+k}\} = A \cup \{2d - k + 1, \dots, n\}$

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Methods, context, and outlook

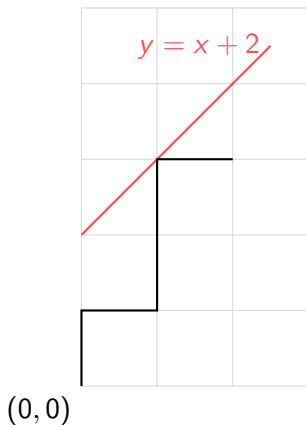


Figure: The (N, E) -lattice path associated to $x_1 x_3 x_4$.

Methods, context, and outlook

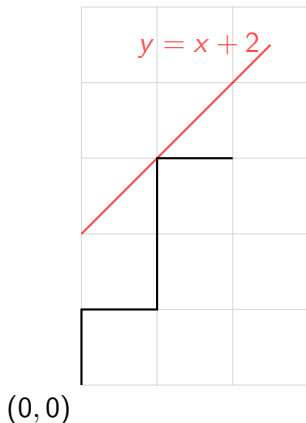


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Remark

We applied our result by...

- ... giving new proof that the squarefree algebra has the strong Lefschetz property
- ... classifying $I_{n,k}$ with regard to the weak Lefschetz property
- ... obtaining minor new result related to Fröberg's conjecture on Hilbert series of complete intersection ideals

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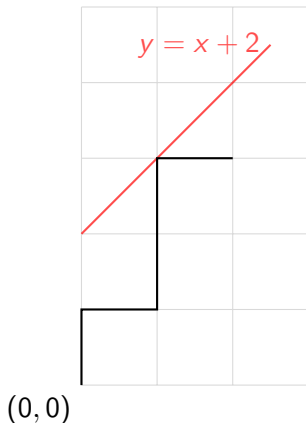


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Outlook

We are working on a generalization to $I_{n,m,k} = (x_1^{m_1}, \dots, x_n^{m_n}, (x_1 + \dots + x_n)^k)$.

Thank you for your attention!